

On the asymptotic methods for nuclear collective models

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Abstract

Contractions of orthogonal groups to Euclidean groups are applied to analytic descriptions of nuclear quantum phase transitions. The semiclassical asymptotic of multipole collective Hamiltonians are also investigated.

I. INTRODUCTION

The notion of Lie algebra and Lie group contractions was first introduced by Segal [1] in 1951 and E. E. Inönü and E. P. Wigner [2] in 1953. The idea of contractions of Lie groups historically appeared in relation with the non-relativistic limit $c \rightarrow \infty$, where c is the speed of light. That limit brings the Poincaré group of relativistic mechanics to the Galilei group of classical mechanics. Then the Euclidean group $E(2)$ is given by the contractions of the groups $SO(3)$ and $SO(1,2)$. Inönü and Wigner have been interested in unitary representations of $E(2)$ in which the spherical harmonics become Bessel functions. The other example is a limit process from quantum mechanics to classical mechanics in the limit $\hbar \rightarrow 0$, where \hbar is the Planck constant. This limit corresponds to the contraction of the Heisenberg algebras to the Abelian ones of the same dimensions. The dynamical symmetries are realized by Lie groups and algebras, superalgebras, quantum groups and algebras. The essential idea of singular and degenerate transformations is presented in all cases of dynamical symmetry contractions. They lead to asymptotic relations between basis functions for the representations of different groups.

In section 2, we study the contractions from the orthogonal group $SO(n+1)$ to the Euclidean group $E(n)$. Moreover, we study the contractions of the corresponding homogeneous spaces: the sphere $S_n \sim O(n+1)/O(n)$ contracts to the Euclidean space $\mathbb{R}^n \sim E(n)/SO(n)$ in the limit $R \rightarrow \infty$, where R is the radius of S_n . It is well known that representations of the Euclidean groups can be constructed in terms of the Bessel functions [3]. We will characterize the contractions from $SO(n+1)$ to $E(n)$ in terms of asymptotic expressions such that the Bessel functions are obtained from the Laguerre functions. We approximate the zeros of the Bessel functions using zeros of Laguerre functions. We obtained this approximation in [4] for the nuclear model $E(5)$ [5]. We apply the results to the dynamical group contractions in the critical points of nuclear phase transition for the models $E(2n+1)$ and $X(2n+1)$.

In section 3, we study the semiclassical spectral properties of collective Hamiltonians. We consider the classical quadrupole collective Hamiltonian \mathcal{H}_n with polynomial potential of degree n . The Henon - Heiles \mathcal{H}_3 is a non-integrable Hamiltonian. Using the Birkhoff-Gustavson normal form method for \mathcal{H}_4 , we explicitly obtain an integrable form as function of two action variables and the corresponding semiclassical quantum spectrum. Moreover, we

consider a spherical multipole quantum Hamiltonian and apply the logarithmic perturbation theory for the radial Schrödinger equation. We obtain the explicit energy spectrum up to the order \hbar^5 .

II. CONTRACTIONS OF ORTHOGONAL GROUPS TO EUCLIDEAN GROUPS

In this section we focus on analytic contractions of orthogonal groups to Euclidean groups.

Let V be a vector space over \mathbb{R} or \mathbb{C} and $f: (0, 1] \rightarrow GL(V)$ be a continuous function. Let $[\cdot, \cdot]$ be a Lie bracket on V . A parametrized family of Lie brackets on V is defined by $[x, y]_\varepsilon = f_\varepsilon([f_\varepsilon^{-1}(x), f_\varepsilon^{-1}(y)])$. If the limit $[x, y]_0 = \lim_{\varepsilon \rightarrow 0} [x, y]_\varepsilon$ exists, then $[\cdot, \cdot]_0$ is a Lie bracket on V and $(V, [\cdot, \cdot]_0)$ is called a *contraction* of $(V, [\cdot, \cdot])$ [6]. For $0 < \varepsilon \leq 1$ the Lie algebras $(V, [\cdot, \cdot]_\varepsilon)$ are all isomorphic to $(V, [\cdot, \cdot])$. Hence to obtain a new Lie algebra via contraction one needs $\det(f_\varepsilon) = 0$ for $\varepsilon = 0$. This is a necessary condition, but not a sufficient one.

We now present the contraction from the orthogonal group $SO(n+1)$ to the Euclidean group $E(n)$. Moreover, we study the contraction from $S_n \sim O(n+1)/O(n)$ to the Euclidean space $\mathbb{R}^n \sim E(n)/SO(n)$ in the limit $R \rightarrow \infty$, where R is the radius of S_n . We shall use R^{-1} as the contraction parameter. To realize the contraction explicitly, let us introduce the Beltrami coordinates on the sphere S_n of radius R as $y_\mu = Ru_\mu/u_0$, $\mu = 1, \dots, n$, with

$$u_0^2 + \sum_{k=1}^n u_k^2 = R^2, \quad (2.1)$$

where u_i , $1 \leq i \leq n$, are real coordinates. The sphere S_n admits the group $SO(n+1)$ of isometries and can be realized as $S_n \sim O(n+1)/O(n)$. Define the differential operators

$$L_{jk} = y_j \partial_{y_k} - y_k \partial_{y_j}, \quad 0 \leq j, k, r, s \leq n, \quad (2.2)$$

with the commutations relations

$$[L_{jk}, L_{rs}] = \delta_{kr} L_{js} + \delta_{js} L_{kr} - \delta_{ks} L_{jr} - \delta_{jr} L_{ks}, \quad 0 \leq j, k, r, s \leq n. \quad (2.3)$$

The basis of the Lie algebra $so(n+1)$ of $SO(n+1)$ consists of L_{jk} , $0 \leq j < k \leq n$, and Laplace-Beltrami operator is given by

$$\Delta_{\text{LB}}^{(n+1)} = \frac{1}{R^2} \sum_{0 \leq j < k \leq n} L_{ij}. \quad (2.4)$$

Introduce now the differential operators

$$q_i = R^{-2} L_{0i} = \partial_{y_i} + R^{-2} \sum_{k=1}^n y_k \partial_{y_k}, \quad 0 \leq i \leq n, \quad (2.5)$$

$$\tilde{L}_{jk} = y_j q_i - y_k q_i = y_j \partial_{y_k} - y_k \partial_{y_j}, \quad 0 \leq j, k \leq n, \quad (2.6)$$

with the commutations relations

$$[\tilde{L}_{jk}, \tilde{L}_{rs}] = \delta_{kr} \tilde{L}_{js} + \delta_{js} \tilde{L}_{kr} - \delta_{ks} \tilde{L}_{jr} - \delta_{jr} \tilde{L}_{ks}, \quad 1 \leq j, k, r, s \leq n, \quad (2.7)$$

$$[q_i, \tilde{L}_{jk}] = \delta_{ij} q_k - \delta_{ik} q_j, \quad [q_j, q_k] = R^{-2} \tilde{L}_{jk}, \quad 1 \leq i, j, k \leq n. \quad (2.8)$$

The Euclidean space \mathbb{R}^n admits the group of isometries $E(n) = \mathbb{R}^n \otimes SO(n)$, where the translation subgroup of the semidirect product is identified to \mathbb{R}^n . Consider now the Killing differential operators

$$p_i = x_i, \quad M_{jk} = x_j \partial_{x_k} - x_k \partial_{x_j}, \quad 1 \leq i, j, k \leq n, \quad (2.9)$$

with the commutations relations

$$[p_i, M_{jk}] = \partial_{ij} p_i - \partial_{ik} p_j, \quad [p_i, p_j] = 0, \quad (2.10)$$

$$[M_{jk}, M_{rs}] = \delta_{kr} L_{js} + \delta_{js} L_{kr} - \delta_{ks} L_{jr} - \delta_{jr} L_{ks}, \quad 1 \leq j, k, r, s \leq n. \quad (2.11)$$

The basis of the Lie algebra $e(n)$ of $E(n)$ consists of p_i and M_{jk} , with $0 \leq i \leq n$ and $0 \leq j < k \leq n$, and Laplace-Beltrami operator is given by

$$\Delta^{(n)} = \sum_{i=1}^n p_i^2. \quad (2.12)$$

For $R \rightarrow \infty$, $SO(n+1)$ and $so(n+1)$ contract to $E(n)$ and $e(n)$, respectively, and $y_i \rightarrow x_i$, $q_i \rightarrow p_i$, $\tilde{L}_{jk} \rightarrow M_{jk}$, and $\Delta_{\text{LB}}^{(n+1)} \rightarrow \Delta^{(n)}$.

When $\nu < -1$, the zeros of J_ν are all real.

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \lim_{n \rightarrow \infty} \frac{1}{n^\nu} L_n^\nu \left(\frac{z^2}{4n}\right), \quad J_\nu(z) = \lim_{\lambda \rightarrow \infty} \lambda^\nu P_\lambda^{-\nu} \left(\cos \frac{z}{\lambda}\right), \quad (2.13)$$

where L_n^ν and $P_\lambda^{-\nu}$, are the Laguerre polynomial and the associated Legendre polynomial, respectively [8].

We now introduce the multipole Hamiltonian

$$H = H_{\text{rad}} + \frac{f(\beta)}{\beta^2} \tilde{H} + V, \quad (2.14)$$

where H_{rad} is the radial Hamiltonian, \tilde{H} is the angular Hamiltonian, $\beta = \sqrt{\sum_{i=1}^{2n+1} x_i^2}$ is the radial variable, and f is an analytic function of β^2 . In particular: 1) $f(\beta) = 1$ for the Bohr model [9] characterized by the group chain $U(2n+1) \supset SO(2n+1) \supset SO(3)$; 2) $f(\beta) = (\beta/\beta_0)^2$, where β_0 is a constant parameter, for the models $E(2n+1)$ and $X(2n+1)$ characterized by the contraction of the orthogonal group $SO(2n+2)$ to the Euclidean group $E(2n+1)$. The Hamiltonian H is separable for $V = V_{\text{rad}} + \tilde{V}f(\beta)/\beta^2$, where V_{rad} is the radial potential Hamiltonian and \tilde{V} is the angular potential. If $V_{\text{rad}} = C\beta^{-2}$, where C is a constant parameter, then the radial Schrödinger equation is a differential equation for the Bessel functions. The $E(2n+1)$ and $X(2n+1)$ models are characterized by an infinite square-well potential in β and the radial energies are proportional to squared zeroes of Bessel functions. According to (2.13), the zeros of the Bessel functions are approximated by the zeros of Laguerre and associated Legendre polynomials. This approximation for the $E(5)$ model is presented in [4]. We obtain an analytic description for transitional nuclei near critical points of quantum phase transitions [10], [11].

III. SEMICLASSICAL ASYMPTOTIC

In this section we study the semiclassical spectral properties of collective Hamiltonians. Consider the following potential of nuclear surface quadrupole oscillations of nuclei [12]:

$$V_n = \sum_{\substack{m, m' \geq 0 \\ 2m+3m' \leq n}} c_{mm'} (q_1^2 + q_2^2)^m (q_1^3 - 3q_1q_2^2)^{m'}. \quad (3.1)$$

Let $q_1 = \beta \cos \gamma$ and $q_2 = \beta \sin \gamma$, where $\beta \geq 0$ and $0 \leq \gamma < 2\pi$. Then

$$V_n = \sum_{\substack{m, m' \geq 0 \\ 2m+3m' \leq n}} c_{mm'} \beta^{2m+3m'} \cos^{m'} 3\gamma. \quad (3.2)$$

Consider the classical Hamiltonian

$$\mathcal{H}_n = \frac{1}{2} (p_1^2 + p_2^2) + V_n, \quad (3.3)$$

where $n \geq 2$. Then \mathcal{H}_3 is the Henon - Heiles Hamiltonian [14].

The critical points of \mathcal{H}_n are given by $\partial V_n/\partial q_1 = \partial V_n/\partial q_2 = 0$. If $c_{10} > 0$, then $q_1 = q_2 = 0$ is a minimum. The solutions of the system of polynomial equations

$$\frac{\partial V_n}{\partial q_1} = 0, \quad \frac{\partial V_n}{\partial q_2} = 0, \quad \frac{\partial^2 V_n}{\partial q_1^2} \frac{\partial^2 V_n}{\partial q_2^2} = \left(\frac{\partial^2 V_n}{\partial q_1 \partial q_2} \right)^2 \quad (3.4)$$

form an algebraic set (separatrix) in the space of the control parameters $c_{mm'}$ and divides it into the regions where V_n is structurally stable. For $n = 4$ consider $c_{10}, c_{20} > 0$ and define $\lambda = 9c_{01}^2/(32c_{10}c_{20})$. If $0 < \lambda < 1$, then there exists a minimum in the origin. The separatrix is given by $\lambda = 1$. In the region $\lambda > 1$ there are four minima and three saddles.

We now define the Birkhoff-Gustavson normal forms [13]. Let $\mathbf{R}^n \times \mathbf{R}^n$ be the phase space endowed with the canonical coordinates (\mathbf{q}, \mathbf{p}) , $\mathbf{q} = (q_1, \dots, q_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$. Let $K(q, p)$ be a Hamiltonian function defined on a domain of $\mathbf{R}^n \times \mathbf{R}^n$ centered in the origin $(\mathbf{0}, \mathbf{0})$, which admits the power-series expansion

$$K(\mathbf{q}, \mathbf{p}) = \sum_{j=1}^n \frac{\nu_j}{2} (p_j^2 + q_j^2) + \sum_{k=3}^{\infty} K_k(\mathbf{q}, \mathbf{p}), \quad (3.5)$$

around $(\mathbf{0}, \mathbf{0})$, where each K_k is a homogeneous polynomial of degree k in (\mathbf{q}, \mathbf{p}) , and ν_j non-vanishing constants. Let (ξ, η) , where $\xi = (\xi_1, \dots, \xi_n)$ and $\eta = (\eta_1, \dots, \eta_n)$, be another canonical coordinates of $\mathbf{R}^n \times \mathbf{R}^n$ and let the power series

$$W(\mathbf{q}, \eta) = \sum_{j=1}^n q_j \eta_j + \sum_{k=3}^{\infty} W_k(\mathbf{q}, \eta) \quad (3.6)$$

be a second type generating function, a function in the old position variables q and the new momentum variables η , associated with the canonical transformation $(\mathbf{q}, \mathbf{p}) \rightarrow (\xi, \eta)$, where $p_j = \partial S/\partial q_j$ and $\xi_j = \partial S/\partial \eta_j$, $1 \leq j \leq n$. Here each W_k is a homogeneous polynomial of degree k in (\mathbf{q}, η) . Then

$$G(\nabla_{\eta} W, \eta) = H(\mathbf{q}, \nabla_{\mathbf{q}} W), \quad (3.7)$$

$$G(\xi, \eta) = \sum_{k \geq 2} G_k(\xi, \eta), \quad G_k(\xi, \eta) = \sum_{j=1}^n \frac{\nu_j}{2} (\eta_j^2 + \xi_j^2), \quad (3.8)$$

where each G_k is a homogeneous polynomial of degree k in (ξ, η) .

The power series $G(\xi, \eta)$ is said to be in the Birkhoff-Gustavson normal form up to degree r if the Poisson commutation relations $\{G_2, G_k\} = 0$ are satisfied for $k = 3, \dots, r$. Here $\{\cdot, \cdot\}$

denotes the canonical Poisson bracket to the coordinates (ξ, η) . We obtain the integrable form

$$K(I_1, I_2) = I_1 + \frac{9c_{01}^2}{24} (I_1^2 - 14I_1I_2 + 14I_2^2) + \frac{c_{20}}{4} (I_1^2 + 2I_1I_2 - 2I_2^2), \quad (3.9)$$

where I_1 and I_2 are action variables. The semiclassical quantum spectrum is given by $K(n_1 + 1/2, n_2 + 1/2)$, where n_1 and n_2 are non-negative integers.

Consider now the spherical multipole Hamiltonian

$$H_N = \frac{1}{2} \sum_{i=1}^{2N+1} p_i^2 + U, \quad U = \sum_{k \geq 1} D_k \beta^{2k}, \quad \beta = \sqrt{\sum_{i=1}^{2N+1} q_i^2}. \quad (3.10)$$

The corresponding classical Hamiltonian is integrable. The radial Schrödinger equation can be written as

$$\left[-\frac{\hbar^2}{2} \frac{d^2}{d\beta^2} + \frac{\hbar^2 l(l+N)}{2\beta^2} + U(\beta) \right] \Phi = E\Phi. \quad (3.11)$$

We apply the logarithmic perturbation theory via the \hbar -expansion technique [15]. Using the substitution, $f(\beta) = \hbar \Phi^{-1}(d\Phi/d\beta)$, the Schrödinger equation (3.11) can be written as a Riccati equation

$$\frac{\hbar}{2} \left(\frac{df}{d\beta} \right)^2 = \frac{\hbar^2 l(l+N)}{2\beta^2} + U(\beta) - E. \quad (3.12)$$

Consider the following series expansions in the Planck constant:

$$E = \sum_{n=0}^{\infty} E_n \hbar^n, \quad f(\beta) = \sum_{n=0}^{\infty} f_n(\beta) \hbar^n. \quad (3.13)$$

The quantization condition

$$\frac{1}{2\pi i} \oint f(r) d\beta = \hbar m, \quad (3.14)$$

where m is the number of zeros of the wave function inside the closed contour, can be rewritten as

$$\frac{1}{2\pi i} \oint f_1(\beta) d\beta = 2n + l + N, \quad \frac{1}{2\pi i} \oint f_n(\beta) d\beta = 0, \quad n > 1, \quad (3.15)$$

where $m = 2n + l + N$. Here n and l are the radial and angular quantum numbers. We

obtain

$$\begin{aligned}
E_1 &= \frac{1}{2} + m, & E_2 &= \frac{1}{2} (3 - 2\lambda + 6v) D_1 \\
E_3 &= \frac{1 + 2m}{8} [(-21 + 9\lambda - 17v) D_1^2 + (15 - 6\lambda + 10v) D_2] \\
E_4 &= \frac{1}{128} [(333 - 201\lambda + 11\lambda^2 - 264\lambda v + 1041v + 375v^2) D_1^3 \\
&\quad - 6 (60 - 39\lambda + 3\lambda^2 - 42\lambda v + 175v + 55v^2) D_1 D_2 \\
&\quad + (105 - 72\lambda + 6\lambda^2 - 60\lambda v + 280v + 70v^2) D_3]
\end{aligned} \tag{3.16}$$

where $v = m(m + 1)$ and $\lambda = l(l + N)$.

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